



Innovative Applications of O.R.

Robust option pricing

Chaithanya Bandi^{a,*}, Dimitris Bertsimas^b^a Operations Research Center, Massachusetts Institute of Technology, 77 Massachusetts Avenue, E40-130, Cambridge, MA 02139, USA^b Operations Research Center, Massachusetts Institute of Technology, 77 Massachusetts Avenue, E40-147, Cambridge, MA 02139, USA

ARTICLE INFO

Article history:

Received 15 February 2011

Accepted 4 June 2014

Available online 2 July 2014

Keywords:

Option pricing
Robust optimization
American option
Volatility smile

ABSTRACT

In this paper, we combine robust optimization and the idea of ϵ -arbitrage to propose a tractable approach to price a wide variety of options. Rather than assuming a probabilistic model for the stock price dynamics, we assume that the conclusions of probability theory, such as the central limit theorem, hold deterministically on the underlying returns. This gives rise to an uncertainty set that the underlying asset returns satisfy. We then formulate the option pricing problem as a robust optimization problem that identifies the portfolio which minimizes the worst case replication error for a given uncertainty set defined on the underlying asset returns. The most significant benefits of our approach are (a) computational tractability illustrated by our ability to price multi-asset, American and Asian options using linear optimization; and thus the computational complexity of our approach scales polynomially with the number of assets and with time to expiry and (b) modeling flexibility illustrated by our ability to model different kinds of options, various levels of risk aversion among investors, transaction costs, shorting constraints and replication via option portfolios.

© 2014 Elsevier B.V. All rights reserved.

1. Introduction

The problem of pricing and hedging derivative securities has been one of the most well studied problems in Financial Economics. The Nobel prize winning contribution was made by Black and Scholes (1973) and Merton (1973) when they used the principle of dynamic replication to obtain a closed form formula for the price of a European option under the assumption that stock returns follow a log-normal distribution with known volatility. The key idea developed in this work is that of dynamic replication where one looks for a portfolio of simpler securities which is self-financing and whose value at the end of the time horizon matches the payoff of the option. Such a portfolio of simpler securities is known as a replicating portfolio, and the value of this portfolio at the beginning of the time horizon, is the no-arbitrage price of the option.

The Black–Scholes model, in spite of its popularity, has some well-known deficiencies. Firstly, a closed form formula is not known for many liquid option classes such as American and Barrier options. This forces one to use computationally expensive simulation based methods to price these options, which do not scale well when the payoff of the option depends on the dynamics of multiple securities, that is when the option is high dimensional. Secondly, there is

ample empirical evidence suggesting that the strong assumption of the underlying asset price following a stationary geometric Brownian motion does not hold. Attempts have been made to model the volatility of the underlying asset as a stochastic quantity. The notable models include Merton's Mixed Jump Diffusion model (Merton, 1976), Cox and Ross' constant elasticity of variance (Cox & Ross, 1976), Hull and White's model (Hull & White, 1987), and Madan, Carr and Chang's variance-gamma model (Madan, Carr, & Chang, 1998). Apart from the problems with the price-dynamics, there are other factors that arise mostly due to institutional rigidities such as transaction costs and liquidity issues, that rule out even the existence of exact replication when markets are incomplete. And even if an exact replication exists, it is not clear how one may compute it in a computationally efficient way.

1.1. Motivation

Motivated by the inability to tractably price different types of options, we look for alternate ways to model the price dynamics that allow computational tractability in high dimensions. In all prior work in asset pricing, the key primitive is the underlying stochastic process for the price dynamics. There are difficulties with assuming a specific stochastic process as the price dynamics:

(a) The only available information is really the returns data. Fitting a specific stochastic process to the data is, in our view, a model of reality, not reality itself.

* Corresponding author.

E-mail addresses: c-bandit@kellogg.northwestern.edu (C. Bandi), dbertsim@mit.edu (D. Bertsimas).

(b) Even if the stochastic process of the underlying price dynamics is known, it may lead to high dimensional dynamic programming that is not computationally tractable even under the ϵ -arbitrage approach, see Bertsimas, Kogan, and Lo (2001).

Given the previous remarks it is, in our view, reasonable to consider alternative modeling approaches that have advantages in terms of tractability.

The second motivation for this work is the success robust optimization has enjoyed in solving high dimensional optimization problems under uncertainty (see Ben-Tal & Nemirovski (1998) and Bertsimas & Sim (2004)). The key philosophical reason behind the success of robust optimization has been the use of uncertainty sets as the underlying model of randomness instead of probability distributions. The resulting robust optimization problems become mathematical optimization models that scale typically polynomially with the dimension of the problem compared to dynamic programming which scale exponentially.

In this paper, we propose to model the underlying price dynamics using polyhedral uncertainty sets. We then use the ϵ -arbitrage approach of Bertsimas et al. (2001) where one seeks a self-financing dynamic portfolio strategy that most closely approximates the payoff of an option. We use the ℓ_1 -norm to measure the error in replication instead of the ℓ_2 -norm used in Bertsimas et al. (2001). This choice of the norm when combined with polyhedral uncertainty sets results in robust linear optimization problems that can be used to price options. Our approach also allows us to easily model transaction costs and other market restrictions such as shorting constraints. Additionally, the use of uncertainty sets allows us to capture very general price dynamics. Furthermore, because the approach results in linear optimization problems, we can accommodate high dimensional problems that, currently, can only be handled by simulation methods. In addition, we adapt our approach to capture the phenomenon of “implied volatility smile” that characterizes the classical Black–Scholes model. Our explanation of the implied volatility smile is that it is caused by different levels of risk aversion of an option writer for different strikes. We model this by constructing different uncertainty sets for different levels of risk-aversion.

1.2. Contributions and paper outline

Our contribution is a proposal to price options that has the following key characteristics:

(a) *Computational Tractability in Pricing High-dimensional Options:* We combine the approach of ϵ -arbitrage replicating portfolios and robust optimization to solve, via linear optimization methods, option pricing models that can model high-dimensional options in markets with transaction costs. We define the dimension of an option as the number of different random variables on which the payoff function depends on. The key advantage here is that unlike Dynamic Programming, our approach scales polynomially (as opposed to exponentially) with the dimension of the option. As evidence of computational tractability and accuracy of the method, we report results for a variety of options (European, Asian, Lookback, American, Index) using empirical data, which show that our approach produces prices that are close to those observed in the options market. Table 1 below summarizes the

number of variables and constraints in the linear optimization model that prices a variety of options.

(b) *Modeling Flexibility:* Our approach allows us to model (a) sophisticated risk measures of the option writer and (b) additional information on return dynamics such as correlations and heavy-tailed behavior by adjusting the uncertainty sets appropriately. For example, in the Central Limit Theorem (CLT) based uncertainty set (7) below, one can adjust the parameter Γ to capture various risk attitudes. As an illustration of the modeling flexibility, we will show how one can capture the “implied volatility smile” that is observed in the market by selecting different Γ 's for different strike prices. Moreover, we can model correlated price dynamics, by appropriately modifying the uncertainty set. Finally our approach allows the ability to price options (a) under transaction costs or other market restrictions, (b) under dividends, and (c) under replication using option portfolios.

The paper is structured as follows. In Section 2, we introduce the overall approach. In Section 3, we use the method to price European call options. In Section 4, we outline how to price Asian, Barrier, Lookback and American options as well as how to handle dividends. In Section 5, we present the method for options in high dimensions. In Section 6, we offer insights on the optimal replicating strategies, while in Section 7, we discuss the modeling flexibility of our approach in modeling the implied volatility smile, transaction costs and replication via option portfolios. Section 8 contains computational results and Section 9 includes our conclusions.

1.3. Notation

Throughout the rest of the paper, we denote scalar quantities by non-bold face symbols (e.g. $x \in \mathbb{R}$, $k \in \mathbb{N}$), vectors and matrices by boldface symbols (e.g. $\mathbf{x} \in \mathbb{R}^n$, $n > 1$ and $\mathbf{A} \in \mathbb{R}^{n \times m}$, $n > 1$, $m > 1$). We denote random variables with a *tilde* (e.g. \tilde{x} , $\tilde{\mathbf{x}}$, $\tilde{\mathbf{A}}$). The bounds on random variables are represented by a bar above or below the symbol representing the random variable (e.g. $\underline{\tilde{x}} \leq \tilde{\mathbf{x}} \leq \bar{\tilde{x}}$).

2. The option pricing problem and price models

An option is a contract defined on a set of predetermined underlying securities, and is associated with a payoff function. The payoff function determines the value of the option after the realization of random returns of the underlying securities. The option pricing problem refers to the problem of calculating the “value” of an option before the realization of the random returns. The payoff function, $P(\{\tilde{S}_\tau^1, \tilde{S}_\tau^2, \dots, \tilde{S}_\tau^M\}, \{K_1, K_2, \dots, K_r\})$, depends on

- (1) $\{\tilde{S}_\tau^1, \tilde{S}_\tau^2, \dots, \tilde{S}_\tau^M\}$: vector of prices of the set of M underlying securities at time τ .
- (2) τ : time at which the option is exercised.
- (3) $\{K_1, K_2, \dots, K_r\}$: a set of other parameters, e.g. strike prices, dividends, etc.

For example, a European Call option's payoff is given by $\max(\tilde{S}_T - K, 0)$ where \tilde{S}_T denotes the price of the underlying security at the time of expiry T , and K denotes the strike price.

Table 1
Number of variables and constraints in the linear optimization problem that prices the options.

Option type	European	Asian	Barrier	Lookback	American	Index	American index
Size	$O(T)$	$O(T)$	$O(T)$	$O(T^2)$	$O(T^2)$	$O(M \cdot T)$	$O(M \cdot T^2)$

T : the number of time periods the option is written for.
 M : the number of different assets required to define the option.
 Size: Number of variables and constraints in our linear optimization model.

To determine the value of the option before the realization of the random returns, we seek to obtain a “replicating portfolio” to capture the payoff dynamics of the option. We construct this portfolio out of the set of stocks and a risk free asset. In this section, we describe how we model the price dynamics of the stocks and how we model this pricing problem as an optimization problem. In this section, we assume the case where the option depends on a single stock.

2.1. Modeling the price dynamics

As described earlier, we do not assume any underlying probability distribution on the returns. Instead we turn to the conclusions of probability theory, especially results that display the concentration of measure phenomenon. We next present examples of uncertainty sets in various cases.

2.1.1. Using historical data and the central limit theorem

Suppose that we have estimated the mean μ and the standard deviation σ of i.i.d. random variables (x_1, \dots, x_n) from historical data. We expect that the central limit theorem holds, and we model uncertainty using the uncertainty set given by

$$U = \left\{ (x_1, \dots, x_n) \mid \left| \sum_{i=1}^n x_i - n\mu \right| \leq \Gamma \sigma \sqrt{n} \right\}. \tag{1}$$

2.1.2. Modeling correlation and ARCH models

Consider the random variables $\mathbf{x} = (x_1, \dots, x_n)$ which are correlated. Specifically, suppose that there are $m < n$ i.i.d. random variables $\mathbf{y} = (y_1, \dots, y_m)$ with mean μ_y and standard deviation σ_y such that $\mathbf{x} = \mathbf{A}\mathbf{y} + \epsilon$, where \mathbf{A} is an $n \times m$ matrix and $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ is a vector of i.i.d. random variables that have mean zero and standard deviation σ_ϵ . Then, we construct the uncertainty set given by

$$U^{\text{corr}} = \left\{ \mathbf{x} \mid \mathbf{x} = \mathbf{A}\mathbf{y} + \epsilon, \left| \sum_{i=1}^m y_i - m\mu_y \right| \leq \Gamma \sigma_y \sqrt{m}, \left| \sum_{i=1}^n \epsilon_i \right| \leq \Gamma \sigma_\epsilon \sqrt{n} \right\}. \tag{2}$$

Using the same approach, we also model autocorrelated returns. For instance, consider an AR(q) model given by

$$y_t = a_0 + a_1 \cdot y_{t-1} + \dots + a_q \cdot y_{t-q} + \epsilon_t,$$

where the return at time t depends on the returns of the previous q periods and ϵ_t 's are i.i.d noise. To model this, we construct the uncertainty set given by

$$U^{\text{AR}(q)} = \left\{ (y_1, \dots, y_T) \mid y_t = a_0 + a_1 \cdot y_{t-1} + \dots + a_q \cdot y_{t-q} + \epsilon_t, \forall t, \left| \sum_{t=1}^T \epsilon_t \right| \leq \Gamma \sigma_\epsilon \sqrt{T} \right\}.$$

2.1.3. Modeling heavy tails

The central limit theorem belongs to a broad class of weak convergence theorems. These theorems express the fact that a sum of many independent random variables tend to be distributed according to one of a small set of stable distributions. When the variance of the variables is finite, the stable distribution is the normal distribution. In particular, these stable laws allow us to construct uncertainty sets for heavy-tailed distributions.

Theorem 2.1 Nolan (1997). Let Y_1, Y_2, \dots be a sequence of i.i.d. random variables, with mean μ and undefined variance. If $Y_i \sim Y$, where Y is a stable distribution with parameter $\alpha \in (0, 2)$, then $(\sum_{i=1}^n Y_i - n\mu/n^{1/\alpha}) \sim Y$.

Motivated by this result, one can construct an uncertainty set U^{HT} representing the random variables $\{Y_i\}$ as follows

$$U^{\text{HT}} = \left\{ (z_1, z_2, \dots, z_n) \mid \left| \sum_{i=1}^n z_i - n\mu \right| \leq \Gamma n^{1/\alpha} \right\}, \tag{3}$$

where Γ can be chosen based on the distributional properties of the random variable Y . Note that U^{HT} is again a polyhedron.

In this paper, we construct uncertainty sets of the form (1) using historical stock price information. In particular, we consider a discrete model of price movements where the price of the stock changes at discrete points of time. Let \tilde{r}_t^S be the return at t ; i.e., the return from period $[t, t + 1)$. Assuming that the returns are identical and independent random variables, we have by applying the CLT to the random variables $\{\tilde{r}_1^S, \tilde{r}_2^S, \dots, \tilde{r}_\tau^S\}$,

$$\frac{\sum_{i=1}^\tau \log(1 + \tilde{r}_i^S) - \tau \cdot \mu_{\log}}{\sigma_{\log} \cdot \sqrt{\tau}} \sim N(0, 1), \text{ as } \tau \rightarrow \infty,$$

where $\mu_{\log}, \sigma_{\log}$ are mean and standard deviation of $\log(1 + \tilde{r}_i^S)$, respectively.

We, therefore, assume as a primitive that this CLT type of behavior happens deterministically. That is, we assume that the returns satisfy

$$\left| \frac{\log \tilde{R}_\tau^S - \tau \cdot \mu_{\log}}{\sigma_{\log} \cdot \sqrt{\tau}} \right| \leq \Gamma \quad \forall \tau, \tag{4}$$

where $\tilde{R}_\tau^S = \prod_{i=1}^\tau (1 + \tilde{r}_i^S)$, is the cumulative return up to time τ , and the parameter Γ determines how conservative or risk averse we want the solution to be. For example, if we select $\Gamma = 2$, then from the CLT we have that the probability of the event described in Eq. (4) is approximately 95%. In Section 7, we discuss further how to choose the parameter Γ .

Note that Eq. (4) is equivalent to

$$\exp(\tau \mu_{\log} - \Gamma \cdot \sqrt{\tau} \cdot \sigma_{\log}) \leq \tilde{R}_\tau^S \leq \exp(\tau \mu_{\log} + \Gamma \cdot \sqrt{\tau} \cdot \sigma_{\log}) \quad \forall \tau, \tag{5}$$

which defines a box uncertainty set for the cumulative returns. In addition, we assume some bounds on the single period return \tilde{r}_τ^S and since $1 + \tilde{r}_\tau^S = \tilde{R}_\tau^S / \tilde{R}_{\tau-1}^S$, we have:

$$\mu_r - \Gamma \sigma_r \leq \frac{\tilde{R}_\tau^S}{\tilde{R}_{\tau-1}^S} \leq \mu_r + \Gamma \sigma_r \quad \forall \tau, \tag{6}$$

where μ_r and σ_r are the mean and standard deviation of $(1 + \tilde{r}_\tau)$, respectively.

In summary, we assume that the cumulative stock returns belong to the following uncertainty set (a polyhedron) defined by (5) and (6)

$$U^{\text{CLT}} = \left\{ \tilde{R}_t^S \mid \underline{\tilde{R}}_t^S \leq \tilde{R}_t^S \leq \overline{\tilde{R}}_t^S, \underline{r}_t^S \cdot \tilde{R}_{t-1}^S \leq \tilde{R}_t^S \leq \overline{r}_t^S \cdot \tilde{R}_{t-1}^S, \forall t = 1, \dots, T \right\}. \tag{7}$$

where $\underline{\tilde{R}}_t^S = e^{t \cdot \mu_{\log} - \Gamma \cdot \sqrt{t} \cdot \sigma_{\log}}, \overline{\tilde{R}}_t^S = e^{t \cdot \mu_{\log} + \Gamma \cdot \sqrt{t} \cdot \sigma_{\log}}, \underline{r}_t^S = \mu_r - \Gamma \cdot \sigma_r, \overline{r}_t^S = \mu_r + \Gamma \cdot \sigma_r$. The values of $\mu_r, \sigma_r, \mu_{\log}, \sigma_{\log}$ can be obtained from empirical data on single period returns. Note that the uncertainty set U^{CLT} is described by $O(T)$ constraints.

2.2. The option pricing problem as an optimization problem

The idea of our approach is to find a replicating portfolio that consists of the underlying stock \mathbb{S} and a risk-free asset \mathbb{B} so that the value of this portfolio at the time of exercise matches the payoff of the option as closely as possible. We refer to the difference as the replication error, which is given by $|P(S_T, K) - W_T|$, where W_T is the value of the portfolio at the time of exercise T . In a robust optimization setting, our goal is to find a portfolio that minimizes

the worst case replication error (denoted by ϵ), between the portfolio wealth and the option payoff, over all possible returns that lie in a predetermined uncertainty set \mathcal{U}^{CLT} defined in Section 2.1. The optimal portfolio thus obtained, will have payoff that is within $\pm\epsilon$ from the actual option payoff for all possible realizations of the returns lying in \mathcal{U}^{CLT} . The price of the option would thus be the initial value of this replicating portfolio.

The associated optimization problem can be represented as follows

$$\begin{aligned} \min_{\{x_t^S, x_t^B, y_t\}} \max_{\{\tilde{R}_t^S \in \mathcal{U}^{\text{CLT}}\}} & |P(S_T, K) - W_T| \\ \text{s.t.} & W_T = x_T^S + x_T^B \\ & x_t^S = (1 + \tilde{r}_{t-1}^S)(x_{t-1}^S + y_{t-1}), \quad \forall t = 1, \dots, T, \\ & x_t^B = (1 + r_{t-1}^B)(x_{t-1}^B - y_{t-1}), \quad \forall t = 1, \dots, T, \end{aligned} \quad (8)$$

where x_t^S is the amount invested in the underlying security, x_t^B is the amount invested in the risk-less asset, and y_t is the amount traded from the underlying security to the risk-less asset during the period $[t, t + 1)$. In this optimization formulation, we seek to minimize the worst case replication error. After finding the portfolio, the price of the option would then be given by $x_0^S + x_0^B$, which is the value of the portfolio at time $t = 0$.

2.2.1. Put-call parity

The optimal replicating portfolio obtained by solving the optimization problem (8) will have a payoff that is within $\pm\epsilon$ from the actual option payoff for all possible realizations of the returns lying in \mathcal{U}^{CLT} . Moreover, as $T \rightarrow \infty$, and assuming complete markets, we are guaranteed to have a replication error of 0. For finite T , we obtain non-zero replication errors but are often close to zero, see Bertsimas, Kogan, and Lo (2000) for a thorough discussion. In the case of incomplete markets and markets with frictions, however, the replication error ϵ is non-zero and can be seen as a measure of incompleteness in the markets.

Model free properties such as the put-call parity defines a relationship between the price of a European call (C) and put (P) option with identical strike (K) prices and expiry (T) in a frictionless market

$$C - P = S_0 - Ke^{-rT}.$$

We next present the put-call parity in the robust framework (the proof is in the Supplementary materials section).

Theorem 2.2. *If ϵ_C and ϵ_P are the replication errors obtained by solving the corresponding option pricing optimization problems (8) for calls and puts respectively, then*

$$S_0 - Ke^{-rT} - \epsilon_C - \epsilon_P \leq C - P \leq S_0 - Ke^{-rT} + \epsilon_C + \epsilon_P. \quad (9)$$

3. Pricing European options

In this section, we present our approach in the context of a European call option that gives the option holder the right to buy the stock at a predetermined price K , at T . and has payoff $P(\tilde{S}_T, K) = (\tilde{S}_T - K)^+$. Using the same set of decision variables and data as in (8), the resulting optimization problem becomes

$$\begin{aligned} \min_{\{x_t^S, x_t^B, y_t\}} \max_{\{\tilde{R}_t^S \in \mathcal{U}^{\text{CLT}}\}} & \left| (\tilde{S}_T - K)^+ - (x_T^S + x_T^B) \right| \\ \text{s.t.} & x_t^S = (1 + \tilde{r}_{t-1}^S)(x_{t-1}^S + y_{t-1}), \quad \forall t = 1, \dots, T, \\ & x_t^B = (1 + r_{t-1}^B)(x_{t-1}^B - y_{t-1}), \quad \forall t = 1, \dots, T. \end{aligned} \quad (10)$$

We next reformulate this robust optimization problem into a linear optimization problem. To do this, we introduce the following variable transformations:

$$\begin{aligned} \alpha_t^S &= \frac{x_t^S}{R_t^S}, \quad \alpha_t^B = \frac{x_t^B}{R_t^B}, \quad \beta_t = \frac{y_t}{R_t^S}, \quad \text{where } \tilde{R}_t^S \\ &= \prod_{i=0}^{t-1} (1 + \tilde{r}_i^S), \quad \text{and } R_t^B = \prod_{i=0}^{t-1} (1 + r_i^B). \end{aligned} \quad (11)$$

Substituting these new variables, we obtain the following formulation:

$$\begin{aligned} \min_{\{\alpha_t^S, \alpha_t^B, \beta_t\}} \max_{\{\tilde{R}_t^S \in \mathcal{U}^{\text{CLT}}\}} & \left| (S_0 \tilde{R}_T^S - K)^+ - (\tilde{R}_T^S \alpha_T^S + R_T^B \alpha_T^B) \right| \\ \text{s.t.} & \alpha_t^S = \alpha_{t-1}^S + \beta_{t-1}, \quad \forall t = 1, \dots, T, \\ & \alpha_t^B = \alpha_{t-1}^B - \beta_{t-1} \frac{\tilde{R}_{t-1}^S}{R_{t-1}^B}, \quad \forall t = 1, \dots, T. \end{aligned}$$

Substituting all intermediate α_t^B , we obtain

$$\min_{\{\alpha_t^S, \alpha_t^B, \beta_t\}} \max_{\{\tilde{R}_t^S \in \mathcal{U}^{\text{CLT}}\}} \left| (S_0 \tilde{R}_T^S - K)^+ - \left(\alpha_0^S + \sum_{t=1}^T \beta_{t-1} \right) \tilde{R}_T^S - \alpha_0^B R_T^B + \sum_{t=0}^{T-1} \beta_t \frac{R_t^B}{R_t^S} \tilde{R}_T^S \right|. \quad (12)$$

We next describe the steps involved in obtaining a linear optimization formulation of (12). The same steps would be used in future sections to obtain linear formulations for other options. Consider the inner optimization problem of (12), and let δ denote its objective value. We observe that δ is the optimal solution of the problem

$$\begin{aligned} \min & \kappa \\ \text{s.t.} & \kappa \geq (S_0 \tilde{R}_T^S - K)^+ - \left(\alpha_0^S + \sum_{t=1}^T \beta_{t-1} \right) \tilde{R}_T^S - \alpha_0^B R_T^B + \sum_{t=0}^{T-1} \beta_t \frac{R_t^B}{R_t^S} \tilde{R}_T^S, \quad \forall \tilde{R}_t^S \in \mathcal{U}^{\text{CLT}}, \\ & \kappa \geq - \left((S_0 \tilde{R}_T^S - K)^+ - \left(\alpha_0^S + \sum_{t=1}^T \beta_{t-1} \right) \tilde{R}_T^S - \alpha_0^B R_T^B + \sum_{t=0}^{T-1} \beta_t \frac{R_t^B}{R_t^S} \tilde{R}_T^S \right), \quad \forall \tilde{R}_t^S \in \mathcal{U}^{\text{CLT}}. \end{aligned}$$

Moreover, in order to capture the piecewise-linear nature of the payoff function $(S_0 \tilde{R}_T^S - K)^+$, we partition the uncertainty set \mathcal{U}^{CLT} according to whether $\tilde{R}_T^S \geq K/S_0$. In particular, let

$$\mathcal{U}_a^1 = \mathcal{U}^{\text{CLT}} \cap \left\{ \tilde{R}_T^S \geq \frac{K}{S_0} \right\}, \quad \mathcal{U}_b^1 = \mathcal{U}^{\text{CLT}} \cap \left\{ \tilde{R}_T^S \leq \frac{K}{S_0} \right\}.$$

Using this partition, we next obtain another equivalent formulation of (12):

$$\begin{aligned} \min_{\{\alpha_0^S, \alpha_0^B, \beta_t\}} & \epsilon \\ \text{s.t.} & \epsilon \geq (S_0 \tilde{R}_T^S - K) - \left(\alpha_0^S + \sum_{t=0}^{T-1} \beta_t \right) \tilde{R}_T^S - \alpha_0^B R_T^B + \sum_{t=0}^{T-1} \beta_t \frac{R_t^B}{R_t^S} \tilde{R}_T^S, \quad \forall \tilde{R}_T^S \in \mathcal{U}_a^1, \\ & \epsilon \geq - \left((S_0 \tilde{R}_T^S - K) - \left(\alpha_0^S + \sum_{t=0}^{T-1} \beta_t \right) \tilde{R}_T^S - \alpha_0^B R_T^B + \sum_{t=0}^{T-1} \beta_t \frac{R_t^B}{R_t^S} \tilde{R}_T^S \right), \quad \forall \tilde{R}_T^S \in \mathcal{U}_b^1, \\ & \epsilon \geq - \left(\alpha_0^S + \sum_{t=0}^{T-1} \beta_t \right) \tilde{R}_T^S - \alpha_0^B R_T^B + \sum_{t=0}^{T-1} \beta_t \frac{R_t^B}{R_t^S} \tilde{R}_T^S, \quad \forall \tilde{R}_T^S \in \mathcal{U}_a^1, \\ & \epsilon \geq - \left(- \left(\alpha_0^S + \sum_{t=0}^{T-1} \beta_t \right) \tilde{R}_T^S - \alpha_0^B R_T^B + \sum_{t=0}^{T-1} \beta_t \frac{R_t^B}{R_t^S} \tilde{R}_T^S \right), \quad \forall \tilde{R}_T^S \in \mathcal{U}_b^1. \end{aligned} \quad (13)$$

The above formulation can be converted to an equivalent linear optimization problem using duality theory as in Bertsimas and Sim (2004) as follows:

$$\begin{aligned} \min & \epsilon \\ \text{s.t.} & \frac{K}{S_0} z_1 + \sum_{t=1}^T \tilde{R}_t^S p_{t,1} + \sum_{t=1}^T \tilde{R}_t^S q_{t,1} - K - \alpha_0^B R_T^B \leq \epsilon, \\ & p_{1,1} + q_{1,1} - r_1^S m_{2,1} - \tilde{r}_1^S n_{2,1} - \beta_1 \frac{R_1^B}{R_1^S} = 0, \\ & p_{t,1} + q_{t,1} + m_{t,1} + n_{t,1} - r_t^S m_{t+1,1} - \tilde{r}_t^S n_{t+1,1} - \beta_t \frac{R_t^B}{R_t^S} = 0, \quad \forall t = 2, \dots, T-1, \\ & z_1 + p_{T,1} + q_{T,1} + m_{T,1} + n_{T,1} - \left(S_0 - \alpha_0^S - \sum_{t=1}^T \beta_{t-1} \right) = 0, \\ & \frac{K}{S_0} z_2 + \sum_{t=1}^T \tilde{R}_t^S p_{t,2} + \sum_{t=1}^T \tilde{R}_t^S q_{t,2} + K + \alpha_0^B R_T^B \leq \epsilon, \\ & p_{1,2} + q_{1,2} - r_1^S m_{2,2} - \tilde{r}_1^S n_{2,2} + \beta_1 \frac{R_1^B}{R_1^S} = 0, \\ & p_{t,2} + q_{t,2} + m_{t,2} + n_{t,2} - r_t^S m_{t+1,2} - \tilde{r}_t^S n_{t+1,2} + \beta_t \frac{R_t^B}{R_t^S} = 0, \quad \forall t = 2, \dots, T-1, \end{aligned}$$

$$\begin{aligned}
 z_2 + p_{T,2} + q_{T,2} + m_{T,2} + n_{T,2} + \left(S_0 - \alpha_0^S - \sum_{t=1}^T \beta_{t-1} \right) &= 0, \\
 \frac{K}{S_0} z_3 + \sum_{t=1}^T \underline{R}_t^S p_{t,3} + \sum_{t=1}^T \overline{R}_t^S q_{t,3} - \alpha_0^B R_T^B &\leq \epsilon, \\
 p_{1,3} + q_{1,3} - \underline{r}_1^S m_{2,3} + \overline{r}_1^S n_{2,3} - \beta_1 \frac{R_T^B}{R_1^B} &= 0, \\
 p_{t,3} + q_{t,3} + m_{t,3} + n_{t,3} - \underline{r}_t^S m_{t+1,3} + \overline{r}_t^S n_{t+1,3} - \beta_t \frac{R_T^B}{R_t^B} &= 0, \quad \forall t \\
 &= 2, \dots, T-1, \\
 z_3 + p_{T,3} + q_{T,3} + m_{T,3} + n_{T,3} + \alpha_0^S + \sum_{t=1}^T \beta_{t-1} &= 0, \\
 \frac{K}{S_0} z_4 + \sum_{t=1}^T \underline{R}_t^S p_{t,4} + \sum_{t=1}^T \overline{R}_t^S q_{t,4} + \alpha_0^B R_T^B &\leq \epsilon, \\
 p_{1,4} + q_{1,4} - \underline{r}_1^S m_{2,4} + \overline{r}_1^S n_{2,4} + \beta_1 \frac{R_T^B}{R_1^B} &= 0, \\
 p_{t,4} + q_{t,4} + m_{t,4} + n_{t,4} - \underline{r}_t^S m_{t+1,4} + \overline{r}_t^S n_{t+1,4} + \beta_t \frac{R_T^B}{R_t^B} &= 0, \quad \forall t \\
 &= 2, \dots, T-1, \\
 z_4 + p_{T,4} + q_{T,4} + m_{T,4} + n_{T,4} - \left(\alpha_0^S + \sum_{t=1}^T \beta_{t-1} \right) &= 0, \\
 p_{t,i} \leq 0, \quad q_{t,i} \geq 0, \quad m_{t,i} \leq 0, \quad n_{t,i} \geq 0, \quad \forall t = 1, \dots, T, \quad \forall i = 1, \dots, 4, \\
 z_1 \leq 0, \quad z_2 \leq 0, \quad z_3 \geq 0, \quad z_4 \geq 0.
 \end{aligned}$$

The size of the linear optimization problem, thus obtained, scales linearly with the number of periods T with $16T + 4$ decision variables and $4T + 4$ constraints.

4. Pricing other single asset options

In this section, we extend our proposed methodology to price other options whose underlying is a single asset. In particular, we discuss the Asian, Barrier, Lookback, American Put Option and how to handle dividends. We discuss how the pricing problems of these options can be formulated as linear optimization problems.

4.1. Pricing Asian options

An Asian option (also called an average option) is an option whose payoff is linked to the average value of the underlying securities on a specific set of dates during the life of the option. Consider an Asian option with an expiry date T and strike price K . If we discretize time into T time periods, the payoff of an Asian call option is given by $(S_{ave} - K)^+$, where $S_{ave} = \sum_{t=1}^T S_t / T$. Here S_t is the price of the security observed at time t . Given this payoff function, we can formulate the pricing problem proceeding as in Section 3. The optimization problem in this case is given by

$$\begin{aligned}
 \min_{\{\alpha_t^S, \alpha_t^B, \beta_t\}} \max_{\{\tilde{R}_t^S \in \mathcal{U}^{CLT}\}} & \left| \left(S_0 \sum_{t=1}^T \frac{\tilde{R}_t^S}{T} - K \right)^+ - \left(\tilde{R}_T^S \alpha_T^S + R_T^B \alpha_T^B \right) \right| \\
 \text{s.t.} \quad \alpha_t^S &= \alpha_{t-1}^S + \beta_{t-1}, \quad \forall t = 1, \dots, T, \\
 \alpha_t^B &= \alpha_{t-1}^B - \beta_{t-1} \frac{\tilde{R}_{t-1}^S}{R_{t-1}^B}, \quad \forall t = 1, \dots, T.
 \end{aligned}$$

We reduce this optimization problem into

$$\begin{aligned}
 \min_{\{\alpha_0^S, \alpha_0^B, \beta_0\}} \epsilon & \\
 \text{s.t.} \quad \epsilon &\geq \left| \left(S_0 \sum_{t=1}^T \frac{\tilde{R}_t^S}{T} - K \right) - \alpha_0^S \tilde{R}_T^S - \alpha_0^B R_T^B + \sum_{t=1}^T \beta_{t-1} \left(\frac{R_T^B}{R_{t-1}^B} \tilde{R}_{t-1}^S - \tilde{R}_t^S \right) \right|, \quad \forall \tilde{R}_t^S \in \mathcal{U}_c^1, \\
 \epsilon &\geq \left| -\alpha_0^S \tilde{R}_T^S - \alpha_0^B R_T^B + \sum_{t=1}^T \beta_{t-1} \left(\frac{R_T^B}{R_{t-1}^B} \tilde{R}_{t-1}^S - \tilde{R}_t^S \right) \right|, \quad \forall \tilde{R}_t^S \in \mathcal{U}_d^1.
 \end{aligned}$$

where \mathcal{U}_c^1 and \mathcal{U}_d^1 define a partition of \mathcal{U}^{CLT} as before. In particular,

$$\mathcal{U}_c^1 = \mathcal{U}^{CLT} \cap \left\{ \sum_{t=1}^T \frac{\tilde{R}_t^S}{T} \geq \frac{K}{S_0} \right\} \text{ and } \mathcal{U}_d^1 = \mathcal{U}^{CLT} \cap \left\{ \sum_{t=1}^T \frac{\tilde{R}_t^S}{T} \leq \frac{K}{S_0} \right\}.$$

Proceeding as in the case of European Options, we construct an equivalent linear optimization problem of size $\mathcal{O}(T)$, which is presented in the [Supplementary materials](#) section.

4.2. Pricing barrier options

Barrier options are contracts whose payoff depends on whether or not the price of the underlying asset crosses a certain level during the option's lifetime. For example, a down-and-out barrier option is active as long as the price of the underlying asset did not go below a predetermined barrier during the option lifetime.

In our framework, all these options can be modeled by appropriately changing the uncertainty sets. Note that these options become inactive as soon as some condition \mathbb{C} of the form $S_t \leq a$ or $S_t \geq b$ is reached. These conditions can be equivalently expressed as linear constraints on the cumulative returns of the form

$$R_t^S \leq a' \text{ or } R_t^S \geq b',$$

appropriately. The problem of optimal replication then reduces to

$$\min_{\{x_t^S, x_t^B, y_t\}} \max_{\{\tilde{R}_t^S \in \mathcal{U}^{CLT} \cap \mathbb{C}\}} |P(S_T, K) - W_T|,$$

because if \mathbb{C} is not satisfied, the option ceases to exist and one need not worry about replicating its payoff. For example, in the case of down-and-out barrier option, the condition \mathbb{C} is given by

$$\tilde{R}_t^S \geq lb, \quad \forall t = 1, \dots, T,$$

where lb is the barrier below which the option is inactive. Therefore the optimal replication problem for this option takes the form of optimization problem (19) with the uncertainty set \mathcal{U}^{CLT} replaced by $\mathcal{U}^{\text{barrier}}$, where

$$\mathcal{U}^{\text{barrier}} = \mathcal{U}^{CLT} \cap \left\{ \tilde{R}_t^S \geq lb, \quad \forall t = 1, \dots, T \right\}.$$

Other barrier options can be modeled in a similar manner. A linear optimization formulation can be obtained following the same approach as before, whose size is again linear in T , which is presented in the [Supplementary materials](#) section.

4.3. Pricing Lookback options

A lookback option is a path dependent option settled based upon the maximum or minimum underlying value achieved during the entire life of the option. At expiration, the holder can look back over the life of the option and exercise based upon the optimal underlying value achieved during that period. Here we consider a fixed strike Lookback call option. Such an option is described by the strike price K and the time of exercise T , with the payoff function given by $(S_0 \tilde{R}_{\max} - K)^+$ where $\tilde{R}_{\max} = \max_{t=1, \dots, T} \{ \tilde{R}_t^S \}$.

Note that the constraint $\tilde{R}_{\max} = \max_{t=1, \dots, T} \{ \tilde{R}_t^S \}$ is non-linear. In order to obtain a linear formulation, we consider a partition $\{\mathcal{U}_k^1\}_{k=1}^T$ of the uncertainty set \mathcal{U}^{CLT} , where

$$\mathcal{U}_k^1 = \mathcal{U}^{CLT} \cap \left\{ R_k^S = R_{\max} \right\} = \mathcal{U}^{CLT} \cap \left\{ R_k^S \geq R_t^S \quad \forall t = 1, \dots, T \right\}.$$

Using this partition, we are able to isolate the sample paths in which R_{\max} occurs in the k th time period, for each $k = 1, \dots, T$. For each of these sets, we obtain the best replicating portfolio and then choose the best among them. This is modeled as follows:

$$\begin{aligned} \min_{\{\alpha_0^S, \alpha_0^B, \beta_t\}} \quad & \epsilon \tag{14} \\ \text{s.t.} \quad & \forall k = 1, \dots, T \\ & \epsilon \geq \left| S_0 \tilde{R}_k^S - K - \alpha_0^S \tilde{R}_T^S - \alpha_0^B R_T^B + \sum_{t=1}^T \beta_{t-1} \left(\frac{R_t^B}{R_{t-1}^B} \tilde{R}_{t-1}^S - \tilde{R}_t^S \right) \right|, \quad \forall \mathcal{U}_k^1 \cap \left\{ \tilde{R}_k^S \geq \frac{K}{S_0} \right\}, \\ & \epsilon \geq \left| - \left(\alpha_0^S + \sum_{t=1}^T \beta_{t-1} \right) \tilde{R}_T^S - \alpha_0^B R_T^B + \sum_{t=1}^T \beta_{t-1} \frac{R_t^B}{R_{t-1}^B} \tilde{R}_{t-1}^S \right|, \quad \forall \mathcal{U}_k^1 \cap \left\{ \tilde{R}_k^S \leq \frac{K}{S_0} \right\}. \end{aligned}$$

We then formulate (14) as a linear optimization problem as before, whose size is quadratic in T . Modeling the non-linear constraint $\tilde{R}_{\max} = \max_{t=1, \dots, T} \{ \tilde{R}_t^S \}$ leads to the increase in size. The resulting linear optimization formulation is presented in the [Supplementary materials](#) section.

4.4. Pricing American options

American style options are described by the strike price K and a time of expiry T , whereas the option can be exercised at any time $\tau \in [0, T]$. We discretize time into T time periods. Assuming that the option is exercised at some instant $\tau \in \{1, 2, \dots, T\}$, the payoff at $t = \tau$ is given by $(S_\tau - K)^+$ for the American Call Option or $(K - S_\tau)^+$ for the American Put. Under the assumption of no dividends paid by the stock, it is always optimal for the American Call option holder to exercise at the date of expiry. This property makes the American Call Option equivalent to that of an European Call Option with the same strike K and the same date of expiry T .

On the other hand, a general optimal exercising strategy is not known for the case of American Put Options. The option holder is expected to solve for the exercising strategy that maximizes his risk-weighted utility. Neither the utility function nor the risk appetite of the option holder is necessarily known to the option writer, and hence as in the case of other options, we cannot write down the payoff function even after all the random returns are revealed. Therefore, we decide to be robust with respect to the option holder's exercising date and hence seek to obtain a replicating portfolio achieving minimum replication error for any exercising policy.

Without loss of generality, we assume that the option holder can exercise at any of the time steps $\tau \in \{1, 2, \dots, T\}$. The problem then reduces to determining the dynamic hedging strategy that minimizes the worst case difference between the replicating portfolio and the payoff accounting for all the possible times at which the option holder can exercise his option.

The problem can be formulated as follows:

$$\begin{aligned} \min_{\{\alpha_t^S, \alpha_t^B, \beta_t\}} \quad & \max_{\{\tilde{R}_t^S \in \mathcal{U}^{\text{CLT}}\}} \max_{\tau=1, \dots, T} \left| (K - S_0 \tilde{R}_\tau)^+ - \left(\tilde{R}_\tau^S \alpha_\tau^S + R_\tau^B \alpha_\tau^B \right) \right| \\ \text{s.t.} \quad & \alpha_t^S = \alpha_{t-1}^S + \beta_{t-1}, \quad \forall t = 1, \dots, T, \\ & \alpha_t^B = \alpha_{t-1}^B - \beta_{t-1} \frac{\tilde{R}_{t-1}^S}{R_{t-1}^B}, \quad \forall t = 1, \dots, T, \end{aligned}$$

where τ stands for the time at which the option is exercised. Substituting all intermediate α_t^B , α_t^S , the above formulation reduces to

$$\begin{aligned} \min_{\{\alpha_t^S, \alpha_t^B, \beta_t\}} \quad & \max_{\{\tilde{R}_t^S \in \mathcal{U}^{\text{CLT}}\}} \max_{\tau=1, \dots, T} \left| (K - S_0 \tilde{R}_\tau)^+ - \left(\alpha_0^S + \sum_{t=1}^{\tau} \beta_{t-1} \right) \tilde{R}_\tau^S - \alpha_0^B R_\tau^B \right. \\ & \left. + \sum_{t=1}^{\tau} \beta_{t-1} \frac{R_t^B}{R_{t-1}^B} \tilde{R}_{t-1}^S \right|. \end{aligned}$$

Proceeding as before, one can obtain another equivalent formulation which we present in Eq. (15) below.

$$\begin{aligned} \min_{\{\alpha_0^S, \alpha_0^B, \beta_t\}} \quad & \epsilon \tag{15} \\ \text{s.t.} \quad & \forall k = 1, \dots, T \\ & \epsilon \geq \left| (K - S_0 \tilde{R}_k^S) - \alpha_0^S \tilde{R}_T^S - \alpha_0^B R_T^B + \sum_{t=0}^{T-1} \beta_t \left(\frac{R_t^B}{R_t^B} \tilde{R}_t^S - \tilde{R}_T^S \right) \right| \\ & \forall \tilde{R}_t^S \in \mathcal{U}^{\text{CLT}} \cap \left\{ \tilde{R}_k^S \leq \frac{K}{S_0} \right\}, \\ & \epsilon \geq \left| - \left(\alpha_0^S + \sum_{t=1}^T \beta_{t-1} \right) \tilde{R}_T^S - \alpha_0^B R_T^B + \sum_{t=0}^{T-1} \beta_t \frac{R_t^B}{R_t^B} \tilde{R}_t^S \right| \\ & \forall \tilde{R}_t^S \in \mathcal{U}^{\text{CLT}} \cap \left\{ \tilde{R}_k^S \geq \frac{K}{S_0} \right\}. \end{aligned}$$

In Eq. (15) the case $k = \tau$ refers to the scenario when the option holder exercises the option during the interval $[\tau, \tau + 1)$. We can now obtain a linear formulation as in the case of previous options. Note that there are $4T$ constraints in (15), this leads to $16T^2 + 4T$ decision variables and $4T^2 + 4T$ constraints. The resulting linear optimization formulation is presented in the [Supplementary materials](#) section.

4.5. Incorporating dividends

In all the options considered so far, we have assumed that the stocks do not yield any dividends. We show in this section how to incorporate dividends. In the absence of dividends, the portfolio evolves according to

$$\begin{aligned} x_t^S &= (1 + \tilde{r}_{t-1}^S)(x_{t-1}^S + y_{t-1}), \quad \forall t = 1, \dots, T, \\ x_t^B &= (1 + r_{t-1}^B)(x_{t-1}^B - y_{t-1}), \quad \forall t = 1, \dots, T. \end{aligned}$$

With dividends, there is a cash infusion in each time period leading to the following evolution

$$x_t^B = (1 + r_{t-1}^B)(x_{t-1}^B - y_{t-1} + r_{t-1}^{\text{div}} \cdot x_{t-1}^S), \quad \forall t = 1, \dots, T,$$

where r_{t-1}^{div} is the rate of dividends, which can also be uncertain. The uncertainty in the dividend rates can also be modeled using an uncertainty set, and given its linear form, this extension does not add any extra complexity in our framework. Using techniques of Section 3, we obtain linear formulations to price options in the presence of dividends.

5. Pricing options in high dimensions

In this section, we present the main advantage of our approach and present our methodology for pricing options that depend on M underlying assets. When M is large, such an option is difficult to price using current methods firstly because of the unavailability of an analytic solution and secondly, because of the curse-of-dimensionality which prevents one from using dynamic programming.

Proceeding as before, we seek to obtain the optimal solution of the following optimization problem

$$\begin{aligned} \min_{\{x_t^m, y_t^m\}} \quad & \max_{\{\tilde{r}_t^m\} \in \mathcal{U}^M} \left| P_f(\{S^i\}_{i=1, \dots, M}, K) - W_T \right| \\ \text{s.t.} \quad & W_T = \sum_{m=0}^M x_T^m, \\ & x_t^m = (1 + \tilde{r}_{t-1}^m) \cdot (x_{t-1}^m + y_{t-1}^m) \quad \forall t = 1, \dots, T, \\ & \forall m = 1, \dots, M, \\ & x_t^0 = (1 + \tilde{r}_{t-1}^0) \cdot \left(x_{t-1}^0 - \sum_{m=1}^M y_{t-1}^m \right) \quad \forall t = 1, \dots, T, \end{aligned} \tag{16}$$

where x_t^m is the amount invested in asset m , y_t^m is the amount added to asset m , and \tilde{r}_t^m is the return from asset m during the period $[t, t + 1)$. r_t^0 is the rate of return from risk-free asset during the same period. $P_f(S, K)$ is the payoff function of the option, where $S = \{S^i\}_{i=1, \dots, M}$, and \mathcal{U}^M is the uncertainty set describing the price dynamics of the M assets. See Eq. (17). The price of the option is then given by $p_0 = \sum_{m=0}^M x_0^m$, where $\{x_0^m\}_{m=1, \dots, M}$ is the optimal solution of (16).

5.1. Price dynamics for multiple assets

While modeling the price dynamics of multiple assets, we need to consider the correlation between the assets. The main aim of this section is, thus, to find uncertainty sets that model this correlation between the asset returns.

As in Section 2.1, let \underline{r}_t^m and \overline{r}_t^m be the lower and upper bound for a single period return at time t , for asset m . Also let $\underline{R}_t^m, \overline{R}_t^m$ be the lower and upper bound for cumulative returns of asset m during the time $[0, t]$. Let Σ be the covariance matrix of the single period returns. Given that Σ is symmetric and positive definite, it has a Cholesky decomposition and we can compute matrix $\mathbf{C} = (\Sigma)^{-1/2}$. Also let $\tilde{\mathbf{R}}_t$ be the $M \times 1$ vector with \tilde{R}_t^m as its entries and let $\tilde{R}_t^m = \mathbb{E}[\tilde{R}_t^m]$. In what follows $\|\mathbf{x}\|$ is a general norm of a vector; usual choices include the ℓ_1, ℓ_2 or the ℓ_∞ norm.

We can thus define the uncertainty set \mathcal{U}^M as follows

$$\mathcal{U}^M = \left\{ \tilde{\mathbf{R}}_t \mid \begin{array}{l} \|\mathbf{C}(\tilde{\mathbf{R}}_t - \tilde{\mathbf{R}}_t)\| \leq \Gamma, \\ \underline{r}_t^m \tilde{R}_{t-1}^m \leq \tilde{R}_t^m \leq \overline{r}_t^m \tilde{R}_{t-1}^m \quad \forall t = 2, \dots, T, \quad \forall m = 1, \dots, M, \\ \underline{R}_t^m \leq \tilde{R}_t^m \leq \overline{R}_t^m, \quad \forall t = 1, \dots, T, \quad \forall m = 1, \dots, M. \end{array} \right\}. \tag{17}$$

The constraint $\|\mathbf{C}(\tilde{\mathbf{R}}_t - \tilde{\mathbf{R}}_t)\| \leq \Gamma$ captures the correlation between the asset returns.

5.2. Optimization formulation

The ability to obtain a linear formulation depends on the norm that we choose for the constraint $\|\mathbf{C}(\tilde{\mathbf{R}}_t - \tilde{\mathbf{R}}_t)\| \leq \Gamma$, as the other constraints in \mathcal{U}^M are all linear. A linear formulation is easy to obtain if we choose the ℓ_1 or ℓ_∞ norm. The D-norm introduced in Bertsimas and Sim (2004) and further explored in Bertsimas, Pachamanova, and Sim (2004) is another norm that allows linear formulations. Moreover the D-norm is attractive given its proximity to ℓ_2 norm (see Proposition 3 in Bertsimas et al. (2004)).

The D-norm of a vector $\mathbf{y} \in \mathbb{R}^{n \times 1}$ for some $d \in [1, n]$ is defined as

$$\|\mathbf{y}\|_d = \max_{\{S \cup \{t\} \mid S \subseteq N, |S| \leq [d], t \in N \setminus S\}} \left\{ \left(\sum_{j \in S} |y_j| \right) + (d - [d]) |y_t| \right\}.$$

In our context, D-norm stands for the maximum possible absolute deviation of the random returns from the correlation-weighted mean returns, that occurs when only d of them are allowed to deviate from their mean values.

The D-norm can be rewritten as the solution of an optimization problem as follows

$$\begin{aligned} \|\mathbf{y}\|_d &= \max_{\left\{ \mathbf{u} \mid \sum_{j=1}^n u_j \leq d, 0 \leq u_j \leq 1 \right\}} \sum_{j=1}^n u_j |y_j| \\ &= \min_{\left\{ r, \mathbf{t} \mid r + t_j \geq |y_j|, t_j \geq 0, j = 1, \dots, n, r \geq 0 \right\}} d \cdot r + \sum_{j=1}^n t_j. \end{aligned}$$

The second equality follows by linear optimization strong duality, when r is the dual variable corresponding to the constraint

$\sum_{j=1}^n u_j \leq d$ and \mathbf{t} is the vector of dual variables corresponding to the constraints $u_j \leq 1$. Thus, the constraint $\|\mathbf{C}(\tilde{\mathbf{R}}_t - \tilde{\mathbf{R}}_t)\|_d \leq \Gamma$ is equivalent to the set of constraints

$$\begin{aligned} r + t_m &\geq \left| \mathbf{C}_m^t (\tilde{\mathbf{R}}_t - \tilde{\mathbf{R}}_t) \right| \quad \forall m = 1, \dots, M, \\ d \cdot r + \sum_{m=1}^M t_m &\leq \Gamma, \\ \mathbf{t} &\geq \mathbf{0}, \quad r \geq 0. \end{aligned}$$

where \mathbf{C}_m refers to the vector corresponding to the m th row of \mathbf{C} . Thus, the D-Norm allows us to construct a polyhedral uncertainty set \mathcal{U}^M which can then be used to obtain linear optimization formulations for multi-dimensional options. The resulting linear optimization formulation is presented in the Supplementary materials section.

6. Characterizing the optimal replicating strategies

In this section, we present some properties of the optimal replicating strategies obtained from solving the optimization problems presented so far. In some cases, we will be able to reduce the dimension of the optimization problems and, in other cases, we will be able to obtain a closed form expression for the optimal replicating strategy.

The first property concerns the nature of the optimal replicating portfolio for an European, Asian and Lookback call option.

Proposition 6.1. *In the absence of any restrictions on α_0^B and α_0^S , there exists an optimal replicating strategy of the form $\{\alpha_0^B, \alpha_0^S, \{\beta_t^s\}_{t=0}^{T-1}\} = \{b, s, \{0, 0, \dots, 0\}\}$, i.e., there exists an optimal strategy where no re-balancing is necessary to optimally replicate the payoffs of European, Asian and Lookback options.*

The validity of Proposition 6.1 is established in the Supplementary materials section. Note that Proposition 6.1 allows us to solve for the option price by solving the following optimization problem

$$\min_{\{\alpha_0^B, \alpha_0^S\}} \max_{\{\tilde{R}_t^S \in \mathcal{U}^{CT}\}} \left| P(S_T, K) - \left(\tilde{R}_T^S \alpha_0^S + R_T^B \alpha_0^B \right) \right|,$$

which is a two dimensional optimization problem, and hence easier to solve. However, this property holds only when there are no restrictions on the positions in the stock and the bond. This may not be true in general and the simple nature of the optimal solution will not hold for such cases.

The second and third properties concern the case of pricing deep in the money options and assert the existence of certain choices of Γ that allow an exact replication and a closed form replication strategy for such options.

Proposition 6.2. *For European call options with strike price K , there exists an interval $[a(K), b(K)]$ such that $\forall \Gamma \in [a(K), b(K)]$ the option price is given by $S_0 - \frac{K}{R_T^B}$.*

Proposition 6.3. *For Asian call options with strike price K , there exists an interval $[a(K), b(K)]$ such that $\forall \Gamma \in [a(K), b(K)]$ the option price is given by $(S_0 \sum_{t=1}^T \frac{R_t^B}{T} - K) / R_T^B$.*

The proof of validity of these propositions is presented in the Supplementary materials section. Propositions 6.2 and 6.3 allow us to obtain the price of the option in closed form for certain values of Γ . For deep-in-the-money options, for a large range of Γ , the conditions of Propositions 6.2 and 6.3 are satisfied. Hence, these results allow us to characterize the option prices of such options in a closed form.

7. Modeling flexibility

In Sections 3–5, we have demonstrated the computational tractability of our approach in pricing various kinds of options in high dimensions. In this section, we discuss the other main advantage of our approach – modeling flexibility. In particular, we discuss (a) modeling the implied volatility smile using the risk-aversion parameter Γ , (b) modeling transaction costs, and (c) incorporating options in our replicating portfolio.

7.1. Modeling the implied volatility smile

As the Black–Scholes model became popular, many started using the model to calculate the volatilities in the market from the market prices observed. This quantity known as the *implied volatility* of an option is simply that volatility that makes the model price exactly equal to the observed market price. Each option has a unique implied volatility, and traders started to quote options in terms of implied volatilities (Derman & Kani, 1994). The main reason is that as the underlying asset price changes through the day, the implied volatility does not have to be adjusted as much as the option prices, which change all the time. The implied volatilities started to appear as fundamental quantities associated with an option.

When the implied volatilities are plotted across strike prices for options with the same time to expiration and on the same underlying stock, it was observed that these plots exhibit *smiles* or *smirks*. According to the Black–Scholes formula, the plot should be a flat line because only one volatility parameter governs the underlying stochastic process on which all options are priced. The same holds for European-style options on the U.S. S&P 500 Index, which were flat from the start of their exchange-based trading in April 1986 until the U.S. stock market crash of October 1987. After the crash, however, volatility smiles became skewed; that is, volatility smiles became downward sloping as the strike price increased. Other markets also often exhibit volatility smiles. Toft and Prucyk (1997) and Mayhew (1995) found downward-sloping volatility smiles for individual stock options, although the curves were not nearly as steep as in the index smiles.

Many explanations have been offered to explain the downward sloping and the U-shaped volatility smiles. As noted by Taylor (1994), there is no economic intuition behind many of these explanations. In particular, Taylor critiques the stochastic volatility models and suggests that it does not capture the true dynamics of the smile. We believe that risk attitude of an investor may be a possible explanation for the existence of the smiles. This is supported from the widely noted empirical observation that the phenomenon of volatility smile started appearing after the crash of 1987.

One of the key features of our pricing methodology is the ability to capture the risk attitude of an option writer, when pricing an option. This is achieved with the help of the parameter Γ which characterizes the uncertainty set that is used for pricing. A lower value for Γ implies that the user is willing to take higher risk by ignoring the variability of stock prices. On the other hand, a higher value of Γ indicates that the user seeks a price that will allow him to replicate the payoff of the option for a larger range of stock prices. Therefore, Γ becomes a natural way to express one’s risk aversion.

In tune with the concept of implied volatility, we define the quantity *Implied Coefficient of Risk Aversion* (Γ_{implied}), as the value of the parameter Γ , which when input to our model gives out a value of the price that matches the market price. We observe, from our experiments, that Γ_{implied} behaves a lot like the implied volatility. When plotted against the strike prices, it displays a U-shaped

behavior and downward sloping. This observation can be explained by simply recalling the meaning of the parameters Γ which stand for the risk aversion of the option writer.

In particular, we observe, from our experiments, that Γ_{implied} varies in a near-quadratic manner with K . When we model this quadratic dependence, we observed that the vertex of the parabola lies very close to the spot price S_0 . This suggests that as K moves away from S_0 , Γ_{implied} increases indicating the increase of risk aversion of the option writer as K moves away from S_0 .

In the experiments, we show empirically that a quadratic variation of Γ_{implied} with K/S_0 would be adequate to characterize the risk aversion of an investor towards different strike prices. We use the following function to describe the relationship:

$$\Gamma(K) = \theta_0 + \theta_1 \frac{K - S_0}{S_0} + \theta_2 \left(\frac{K - S_0}{S_0} \right)^2, \quad \theta_2 \geq 0. \tag{18}$$

The quantity $(K - S_0)/S_0$ captures the distance between the strike and the spot price and is also called as *moneyness* in the literature. We use a quadratic regression model to compute the coefficients $\{\theta_0, \theta_1, \theta_2\}$ so that the prices obtained using these Γ ’s match the market prices of our training set of strikes. We then use the resulting quadratic model $\Gamma(K)$ to calculate Γ and input it to yield the price for options with other strike prices.

7.2. Modeling transaction costs and other market constraints

In this section, we show how our framework has the ability to model transaction costs and other market constraints. The optimal replication problem for an option with an arbitrary payoff function is given by (8), and when we account for the transaction costs, the problem changes to

$$\begin{aligned} & \min_{\{x_t^S, x_t^B, y_t\}} \max_{\{\tilde{R}_t^S \in \mathbb{U}^1\}} |P(S_T, K) - (x_T^S + x_T^B)| \\ & \text{s.t. } x_t^S = (1 + \tilde{r}_{t-1}^S)(x_{t-1}^S - u_{t-1} + v_{t-1}), \quad \forall t = 1, \dots, T, \\ & \quad x_t^B = (1 + r_{t-1}^B)(x_{t-1}^B + (1 - c_{\text{sell}})u_{t-1} - (1 + c_{\text{buy}})v_{t-1}), \\ & \quad \forall t = 1, \dots, T, \end{aligned}$$

where u_t is the amount removed from the stock and v_t is the amount added to the stock during the time period $[t, t + 1)$. Using the variable transformations as defined in (11), we obtain the following equivalent formulation

$$\begin{aligned} & \min_{\{\alpha_t^S, \alpha_t^B, \beta_t\}} \max_{\{\tilde{R}_t^S \in \mathbb{U}^1\}} |P(S_T, K) - (\tilde{R}_T^S \alpha_T^S + R_T^B \alpha_T^B)| \\ & \text{s.t. } \alpha_t^S = \alpha_{t-1}^S + \beta_{t-1}^1 - \beta_{t-1}^2, \quad \forall t = 1, \dots, T \\ & \quad \alpha_t^B = \alpha_{t-1}^B + ((1 - c_{\text{sell}})\beta_{t-1}^1 - (1 + c_{\text{buy}})\beta_{t-1}^2) \frac{\tilde{R}_{t-1}^S}{R_{t-1}^B}, \\ & \quad \forall t = 1, \dots, T. \end{aligned}$$

Observing that

$$\begin{aligned} \alpha_T^S &= \alpha_0^S + \sum_{t=1}^T (\beta_{t-1}^1 - \beta_{t-1}^2), \\ \alpha_T^B &= \alpha_0^B + \sum_{t=1}^T \left\{ ((1 - c_{\text{sell}})\beta_{t-1}^1 - (1 + c_{\text{buy}})\beta_{t-1}^2) \frac{\tilde{R}_{t-1}^S}{R_{t-1}^B} \right\}, \end{aligned}$$

we obtain equivalent formulations, which we formulate as linear optimization problems as in Section 3. There have been alternative approaches (see for example Zakamouline (2006a, 2006b)) to the option pricing problem involving transaction costs under Brownian dynamics. To the best of our knowledge, however, these approaches do not extend to non-Brownian dynamics.

We also remark that other market constraints such as limits on shorting and limits on the leverage ratio which can be modeled by linear constraints can also be incorporated in our approach without adding any computational complexity. In general, the proposed approach is capable of handling transaction costs and other real world constraints without an increase in complexity.

7.3. Replication using option portfolios

In Sections 3–5, the replicating portfolios consisted of the underlying stocks and bonds. In this section, we consider portfolios that also includes options. As an illustration, we consider an European Option with strike K , expiry T . We seek to construct a replicating portfolio consisting of the stock, the bond, and m options with strikes $\{K_1, \dots, K_m\}$ with expiry T . Without loss of generality, let $K_1 \leq \dots \leq K_q \leq K \leq K_{q+1} \leq \dots \leq K_m$. Using the notation from Section 3, and let $\{x_t^S, x_t^B\}_{t=1}^T$ denote the positions in stock and bond, and let $\{w_i\}_{i=1}^m$ denote the positions in the m options. The option pricing problem is, thus, given by

$$\min_{\{x_t^S, x_t^B, y_t\}} \max_{\{\tilde{R}_t^S \in \mathcal{U}^{\text{CLT}}\}} \left| \left(\tilde{S}_T - K \right)^+ - \left(x_T^S + x_T^B + \sum_{i=1}^m w_i \left(\tilde{S}_T - K_i \right)^+ \right) \right| \quad (19)$$

s.t. $x_t^S = (1 + r_{t-1}^S)(x_{t-1}^S + y_{t-1}), \quad \forall t = 1, \dots, T,$
 $x_t^B = (1 + r_{t-1}^B)(x_{t-1}^B - y_{t-1}), \quad \forall t = 1, \dots, T.$

We next reformulate this robust optimization problem into a linear optimization problem. Using the variable transformations as defined in (11), we obtain the following equivalent formulation

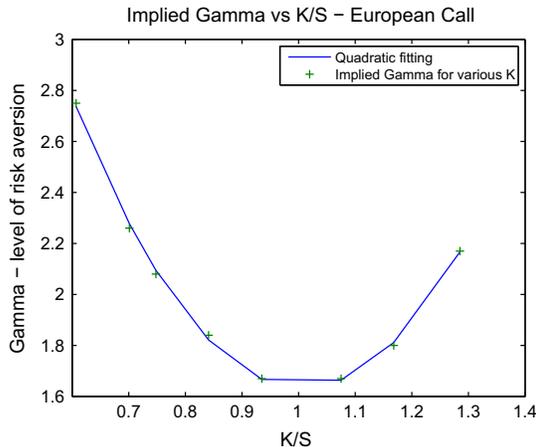
$$\min_{\{\alpha_t^S, \alpha_t^B, \beta_t\}} \max_{\{\tilde{R}_t^S \in \mathcal{U}^{\text{CLT}}\}} \left| \left(S_0 \tilde{R}_T^S - K \right)^+ - \left(\tilde{R}_T^S \alpha_T^S + R_T^B \alpha_T^B + \sum_{i=1}^m w_i \left(S_0 \tilde{R}_T^S - K_i \right)^+ \right) \right|$$

s.t. $\alpha_t^S = \alpha_{t-1}^S + \beta_{t-1}, \quad \forall t = 1, \dots, T,$
 $\alpha_t^B = \alpha_{t-1}^B - \beta_{t-1} \frac{\tilde{R}_{t-1}^S}{R_{t-1}^B}, \quad \forall t = 1, \dots, T.$

Substituting all intermediate α_t^B, α_t^S , we obtain

$$\min_{\{\alpha_0^S, \alpha_0^B, \beta_t\}} \max_{\{\tilde{R}_t^S \in \mathcal{U}^{\text{CLT}}\}} \left| f \left(S_0, K, K_1, \dots, K_m, \tilde{R}_T^S \right) - \left(\alpha_0^S + \sum_{t=1}^T \beta_{t-1} \right) \tilde{R}_T^S - \alpha_0^B R_T^B \right. \\ \left. + \sum_{t=0}^{T-1} \beta_t \frac{R_t^B}{R_t^B} \tilde{R}_t^S \right|, \quad (20)$$

where



$$f \left(S_0, K, K_1, \dots, K_m, \tilde{R}_T^S \right) = \left(S_0 \tilde{R}_T^S - K \right)^+ - \sum_{i=1}^m w_i \left(S_0 \tilde{R}_T^S - K_i \right)^+.$$

In order to capture the piecewise linear nature of the terms $\left(S_0 \tilde{R}_T^S - K_i \right)^+$, we proceed as in Section 3, to define $m + 2$ partitions

$$\left\{ \mathcal{U}_{-\infty, K_1}^1, \mathcal{U}_{K_1, K_2}^1, \dots, \mathcal{U}_{K_q, K}^1, \mathcal{U}_{K, K_{q+1}}^1, \dots, \mathcal{U}_{K_m, \infty}^1 \right\},$$

of the uncertainty set \mathcal{U}^{CLT} , where $\mathcal{U}_{a,b}^1$ is defined as

$$\mathcal{U}_{a,b}^1 = \mathcal{U}^{\text{CLT}} \cap \left\{ \frac{a}{S_0} \leq \tilde{R}_T^S \leq \frac{b}{S_0} \right\}.$$

Using this partition, we obtain an optimization problem along the lines of (13), which we convert into an equivalent linear optimization problem, by using methods in Bertsimas and Sim (2004).

The same procedure can be used for portfolios of other kinds of options. As long as the portfolio of options have piecewise linear payoff functions, we obtain linear optimization formulations for the option pricing problem.

8. Computational results

In this section, we first describe how to use our methodology (a) to price an option, (b) to dynamically hedge at every time step to replicate as closely as possible the payoff of the option. We use our methodology to price various kinds of options described in this paper and compare our results against other methodologies and against prices observed in the market.

8.1. Pricing and hedging methodology

An option-writer would have to solve two related problems (a) Price the option – the pricing problem and (b) Starting with the

Table 2
European call option – price, replication error.

No.	T	K/S	Γ_{implied}	Mkt price	Model price	Error	ϵ
1	18	0.654	2.45	7.475	7.48	0.005	0.1769
2	18	0.794	2.056	4.8	4.797	-0.003	0.6542
3	18	0.888	1.75	3.25	3.232	-0.018	1.0989
4	18	0.981	1.66	1.97	1.968	-0.002	1.3692
5	18	1.028	1.65	1.47	1.462	-0.008	1.5239
6	18	1.121	1.73	0.735	0.749	0.014	1.8572
7	18	1.402	2.96	0.055	0.047	-0.008	2.479

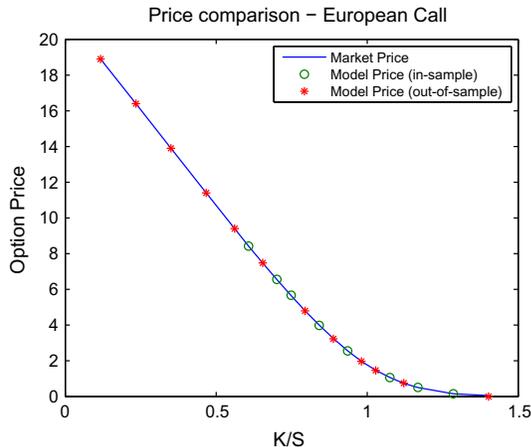


Fig. 1. European call option pricing: coefficients of risk aversion and fitting the market prices.

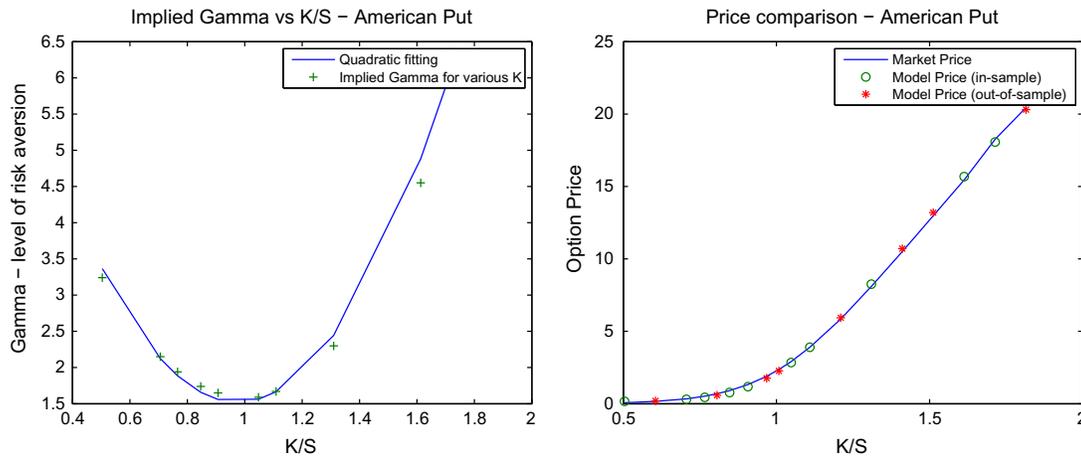


Fig. 2. American put option pricing: coefficients of risk aversion and fitting the market prices.

Table 3
American put option – replication error for $\Gamma(K)$ as in Eq. (18).

No.	T	K/S	$\Gamma_{implied}$	Mkt price	Model price	Error	€
1	25	0.605	2.62	0.17	0.201	0.031	0.3982
2	25	0.806	1.83	0.695	0.589	-0.106	0.9996
3	25	0.968	1.6	1.895	1.764	-0.132	1.9487
4	25	1.008	1.59	2.365	2.266	-0.099	2.2109
5	25	1.21	1.9	5.85	5.939	0.089	3.5781
6	25	1.411	2.87	10.5	10.703	0.203	5.9713
7	25	1.512	3.63	12.975	13.2	0.225	6.832
8	25	1.815	7.7	20.45	20.303	-0.147	12.5842

Table 4
Comparison of replication errors for American Put Option with $\Gamma(K)$ as in Eqs. (18) and (21).

No.	T	K/S	ϵ_1	ϵ_2
1	25	0.605	0.3982	0.237
2	25	0.806	0.9996	0.5342
3	25	0.968	1.9487	1.08
4	25	1.008	2.2109	1.2482
5	25	1.21	3.5781	1.9217
6	25	1.411	5.9713	2.8723
7	25	1.512	6.832	3.2981
8	25	1.815	12.5842	7.1987

price of the option, dynamically hedge to replicate the payoff at the time of exercise – the dynamic hedging problem.

8.1.1. The pricing problem

The pricing problem can be solved by solving the linear optimization problems that we presented in previous sections. This problem needs the following inputs: (1) Option parameters – $\{K, T\}$, (2) Historical data, in particular the means and variances – $\{\mu, \sigma, \mu_{log}, \sigma_{log}\}$, and (3) The Coefficient of Risk Aversion (Γ).

The first two sets of inputs are obtained from the specifications of the option and from historical data, respectively. The other input parameter required to price the option is the coefficient of risk aversion.

8.1.2. The dynamic hedging problem

Once an option writer prices an option and sells it, he then seeks to use the capital obtained from selling the option, to replicate the payoff of the option at the time of exercising. The following algorithm (Algorithm 1) can be used to do this. The inputs to this algo-

gorithm are the price of the option and the option pricing linear optimization problem that one used to price the option.

Algorithm 1. Dynamic Hedging using the Linear Optimization Option Pricer

- **Step 1:** At $t = 0$, Solve the Linear Optimization problem that prices the particular option to obtain the price = $\alpha_0^B + \alpha_0^S$, where α_0^B is the amount invested in the bonds and α_0^S is the amount invested in the stock. Let $b_t = \alpha_0^B$ and $s_t = \alpha_0^S$ for this iteration.
- **While** ($t < T$)
 - Add the constraint $\alpha_0^S + \alpha_0^B = b_{t-1}(1 + r_t^B) + s_{t-1}(1 + r_t^S)$ to the optimization problem (This corresponds to the self-financing nature of our approach).
 - Solve the modified optimization problem to obtain b_t and s_t .
 - Re-balance according to b_t and s_t .

8.2. Experimental results

In this section, our goal is to demonstrate how well we can adapt our methodology (by choosing appropriate values for Γ) to obtain prices that match with the prices that are observed in the market. We perform three experiments, each experiment dealing with a specific type of option. We consider European call options in Experiment 1, American Put options in Experiment 2 and Index call options in Experiment 3. Each experiment consists of a training and a testing stage. In the training stage, we choose a random set of strike prices and calibrate (compute $\theta_0, \theta_1, \theta_2$ in Eq. (18)) our model to match the option prices corresponding to these strikes. In the testing stage, we use the calibrated model to price the options for the remaining strikes.

8.3. Experiment 1: Comparison with market prices for European call options

In this experiment, we aim to price Microsoft (MSFT) 18 week European call options with spot price $S_0 = \$21.4$ for various strikes in the range \$2.5–\$30. In the training stage, we compute $\Gamma_{implied}(K)$ for each of the options with strikes in the training set. Then we fit a quadratic function to these values, thus obtaining the coefficients of the quadratic function in (18). We observe from Fig. 1, that the assumption of quadratic dependence of the Γ 's on the strike prices, is empirically valid. We observe that for deep-in-the-money options, the set of values of Γ that give us the market price is an

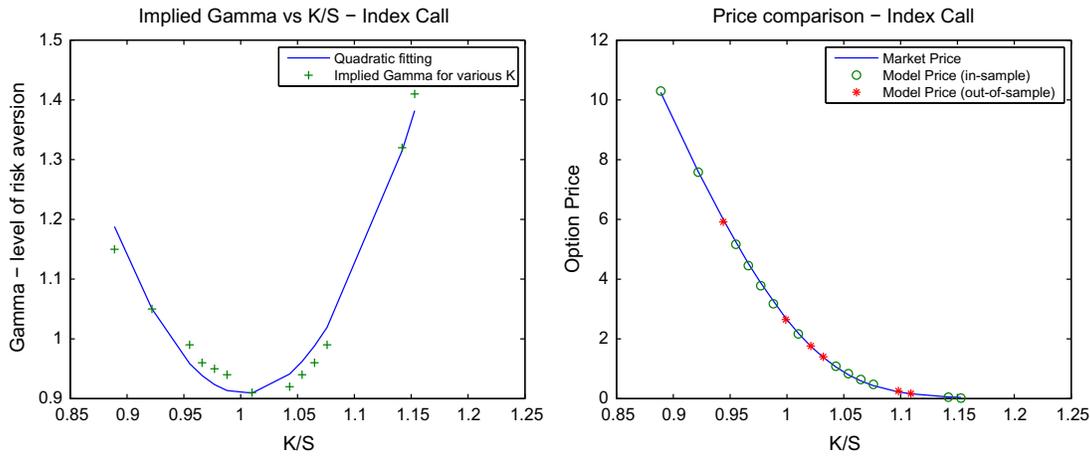


Fig. 3. European index call option pricing: coefficients of risk aversion and fitting the market prices.

Table 5

European index call option – price, replication error.

No.	T	K/S	$\Gamma_{implied}$	Mkt price	Model price	Error	ϵ
1	8	0.944	1.01	6	5.918	-0.082	2.002
2	8	0.999	0.92	2.69	2.647	-0.043	2.5717
3	8	1.021	0.91	1.75	1.762	0.012	2.6012
4	8	1.032	0.92	1.38	1.402	0.022	2.5615
5	8	1.098	1.07	0.225	0.254	0.029	1.9665
6	8	1.109	1.13	0.16	0.17	0.01	1.881

Table 6

Summary: percentage errors between the model prices and the market prices.

Errors\option type	Asian (%)	Lookback (%)	Asian index (%)
Maximum absolute error (in-sample)	12.3	17.1	14.4
Average absolute error (in-sample)	7.4	13.4	10.3
Maximum absolute error (out-of-sample)	18.7	19.3	17.3
Average absolute error (out-of-sample)	11.2	15.3	12.7

Table 7

Computational times for European option pricing problems with replication error $\epsilon < 1\%$ using uncertainty set \mathcal{U}^{CLT} .

Number of assets M	50	100	200	500
Time (in minutes)	0.03	0.1	0.5	1.5

interval, as was explained by Proposition 6.2. The out of sample results are tabulated in Table 2.

8.4. Experiment 2: Comparison with market prices for American put options

In this experiment, we consider MSFT 25 week American Put Options, with spot price $S_0 = \$24.8$ and strike prices in the range \$7.5–\$50. We perform this experiment with real market prices. The results are presented in Fig. 2.

We observe from Table 3 that, for away-from-the-money situations, option writers have larger risk aversion as implied by bigger magnitude of $\Gamma_{implied}$, which is consistent with Eq. (18).

Table 8

Computational times for Asian option pricing problems with replication error $\epsilon < 1\%$ using uncertainty set \mathcal{U}^{CLT} .

Number of assets M	50	100	200	500
Time (in minutes)	0.05	0.25	1.5	2

Table 9

Effect of the uncertainty set on the computational times (in minutes) for European and Asian option pricing problems with replication error $\epsilon < 1\%$.

Uncertainty set	\mathcal{U}^{CLT}	\mathcal{U}^{HT}	\mathcal{U}^{Corr}
European option	1.5	1.6	2.5
Asian option	2	2.2	3

Note however that the replication errors are higher for American options. This is due to the simplistic choice of Eq. (18). By including other option-specific effects or higher order terms in the choice of functional form of $\Gamma(K)$ will provide us with a better fit. For instance, we considered the following different functional form in Eq. (21) for the case of American Put options.

$$\Gamma(K) = \theta_0 + \theta_1 \frac{K - S_0}{S_0} + \theta_2 \left(\frac{K - S_0}{S_0}\right)^2 + \theta_3 \left(\frac{K - S_0}{S_0}\right)^3. \tag{21}$$

In Table 4, we present a comparison of the replication errors ϵ_1 and ϵ_2 obtained by using the functional forms in Eqs. (18) and (21), respectively. We observe that we obtain an improvement of almost 50% for all the cases.

8.5. Experiment 3: Comparison with market prices for European index options

In this experiment, we consider the 1/100 Dow Jones Industrial Average Index Options (DJIA). The underlying value of these options is based on the level of the Dow Jones Industrial Average, a price-weight stock market index calculated from the stock prices of thirty of the largest public companies in the US. We consider 8 week options with spot $S_0 = \$90.8$, for strike prices in the range \$74–\$105.

We again observe from Fig. 3, a quadratic relationship between the $\Gamma_{implied}$ and the strike price K. In tune with our expectations, for in-the-money and out-of-the-money situations, we observe larger

risk aversion as implied by bigger magnitude of T_{implied} . The fit between the prices obtained is also encouraging.

8.6. Summary of other experiments

Table 6 summarizes the results of all the experiments considered and include Asian and Lookback options. The errors between the model prices and the market prices for the single asset European options, and European style Index option are smaller than the errors obtained in other types of options. In general, when the option type is simpler (e.g. European or Asian) the replication error is small. The largest replication error occurs in the American put option where the price we produce tries to protect the holder against the worst choice for exercising the option, thus adding an additional layer of conservativeness.

8.7. Computational times

As discussed in Table 1, the computational complexity of our approach depends on the number of assets M and the trading frequency/discretization T . In particular, for European and Asian style options, the size of the option pricing linear optimization problem is of the order $\mathcal{O}(M \cdot T)$. Moreover, the replication error or the ϵ -arbitrage decreases with increase in discretization T . In order to evaluate the tractability of our approach, we considered pricing M -asset European and Asian options with $M = 200, 500$ using the \mathcal{U}^{CLT} uncertainty set. We calculated the computational times of the option pricing problem for a discretization T chosen to achieve a replication error of within 1%. We present the results in Tables 5 and 6 below. We observe that we were able to solve the optimization problems within a matter of minutes.

We next considered the effect of the uncertainty set on the computational times. In particular, we set $M = 500$ and computed the computational times when we change the uncertainty set from \mathcal{U}^{CLT} to $\mathcal{U}^{\text{Cort}}$ and \mathcal{U}^{Ht} . We present the results in Table 7 and observe that the type of uncertainty set does not have considerable effect on the computational times (see Tables 8 and 9).

9. Conclusions

In this paper, we make a case for the need of an alternate approach to model the uncertainty in the asset returns and propose an uncertainty set based model for the same. This approach combined with the ϵ -arbitrage approach allows us to price a variety of options using linear optimization, even under the presence of transaction costs and multiple assets. The main advantage of our model is that unlike dynamic programming, our approach scales

polynomially (as opposed to exponentially) with the dimension of the original pricing problem. We illustrate our method and report results for a variety of options (European, Asian, Lookback, American, Index). The results show that our approach produces prices that are close to those observed in the options market.

Acknowledgements

We would like to thank the three reviewers of the paper for insightful comments.

Appendix A. Supplementary material

Supplementary data associated with this article can be found, in the online version, at <http://dx.doi.org/10.1016/j.ejor.2014.06.002>.

References

- Ben-Tal, A., & Nemirovski, A. (1998). Robust convex optimization. *Mathematics of Operations Research*, 23, 769–805.
- Bertsimas, D., Kogan, L., & Lo, A. W. (2000). When is time continuous? *Journal of Financial Economics*, 55, 173–204.
- Bertsimas, D., Kogan, L., & Lo, A. W. (2001). Hedging derivative securities and incomplete markets: An ϵ -arbitrage approach. *Operations Research*, 49(7), 372–397.
- Bertsimas, D., Pachamanova, D., & Sim, M. (2004). Robust linear optimization under general norms. *Operations Research Letters*, 32, 510–516.
- Bertsimas, D., & Sim, M. (2004). The price of robustness. *Operations Research*, 52(1), 35–53.
- Black, F., & Scholes, M. (1973). Pricing of options and corporate liabilities. *Journal of Political Economy*, 81, 637–654.
- Cox, J. C., & Ross, S. A. (1976). The valuation for alternative stochastic processes. *Journal of Financial Economics*, 3, 145–166.
- Derman, E., & Kani, I. (1994). Riding on the smile. *RISK*, 7, 32–39.
- Hull, J., & White, A. (1987). The pricing of options on assets with stochastic volatility. *Journal of Finance*, 42, 281–300.
- Madan, D. B., Carr, P. P., & Chang, E. C. (1998). The variance gamma process and option pricing. *European Finance Review*, 2, 79–105.
- Mayhew, S. (1995). Implied volatility. *Financial Analysts' Journal*, 51(4), 8–20.
- Merton, R. C. (1973). An intertemporal capital asset pricing model. *Econometrica*, 41, 867–887.
- Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3, 125–144.
- Nolan, J. (1997). Numerical calculation of stable densities and distribution functions: Heavy tails and highly volatile phenomena. *Communications in Statistics, Stochastic Models*, 13, 759–774.
- Taylor, S. J. (1994). Modelling stochastic volatility: A review and comparative study. *Mathematical Finance*, 4(2), 183–204.
- Toft, K., & Prucyk, B. (1997). Options on leveraged equity: Theory and empirical tests. *Journal of Finance*, 52, 115–180.
- Zakamouline, V. (2006a). Efficient analytic approximation of the optimal hedging strategy for a European call option with transaction costs. *Quantitative Finance*, 6, 435–445.
- Zakamouline, V. (2006b). European option pricing and hedging with both fixed and proportional transaction costs. *Journal of Economic Dynamics and Control*, 30, 1–25.